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DOUBLE-PRICING DUAL AND FEASIBLE START  
ALGORITHMS FOR THE CAPACITATED  
TRANSPORTATION (DISTRIBUTION) PROBLEM

Fred Glover, et al

Texas University

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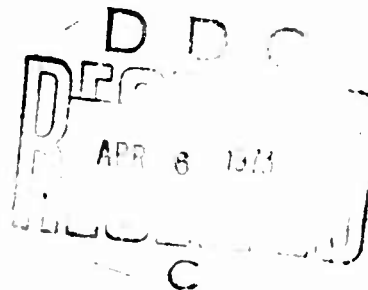
by

Fred Glover\*  
D. Karney\*\*  
D. Klingman\*\*\*

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May 1970

(Revised October 1972)

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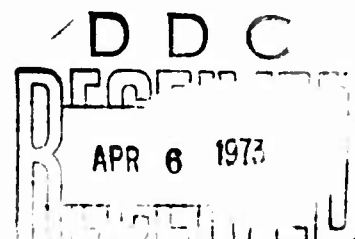
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13. ABSTRACT  The primary objectives of this paper are: (1) to present a simplified "double-pricing" method for solving the capacitated transportation problem by Lemke's dual method which streamlines computer implementation; (2) to give a new and efficient method for obtaining a dual feasible starting basis; (3) to give the results of computational comparison of a code based on these developments with two widely used out-of-kilter production codes. In addition, these codes are compared against a state of the art large scale I.P code, OPHELIE/LP.  The study shows that the improved dual transportation algorithm is faster than the out-of-kilter codes for problems of up to 150 x 150 (origins x destinations), but tends to fall behind thereafter. The best of these algorithms was found to be at least 20 times faster than OPHELIE.			

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## 1.0 INTRODUCTION

This paper proposes a simplified "double-pricing" method for solving the capacitated transportation problem by Lemke's dual algorithm [25] and shows how to streamline this algorithm for computer implementation. We also provide an efficient method for obtaining a dual feasible starting basis that exhibits certain advantages over counterpart methods that have been proposed for obtaining a primal feasible basis. These methods can be applied to network models by using the technique in Wagner [35, page 173] to transform the network into a distribution problem. In addition, the results of computational comparison of a code based on these developments with two widely used out-of-kilter production codes is provided. These codes are also compared against a state of the art LP code, OPHELIE/LP.

Specialized methods for solving the transportation problem with the primal simplex algorithm and with related "dual" or "primal-dual" network algorithms have been the focus of a great deal of ingenuity and effort (See, e.g. [2,3,6,7,9,11,13,14]). A method for exploiting the topological structure of the transportation problem in a dual context was first proposed by Balas and Ivanescu [2,3], and later simplified and shown to constitute a specialization of the dual method by Charnes and Kirby [7]. The key to effecting the simplifications of Charnes and Kirby lay in the use of the Charnes and Cooper "poly-u" procedure [6] for linear programming.

The motivation cited for the dual methods of [2,3,7] was the supposition that the problem's "supplies" and "demands" may not be permanently fixed but subject to change. In such a situation, it is useful to have the ability to begin from an optimal basis to a given problem and proceed via the dual method to an optimal solution for a problem with altered shipment requirements.

However, to our knowledge, no efficient procedure has appeared in the literature for obtaining an initial dual feasible basis for a problem that has not previously been solved under a stated set of shipment requirements. Moreover, the dual methods of [2,3,7] make no provision for accommodating the capacitated transportation problem, in which the variables are constrained to lie within stated bounds. To handle these more general considerations, the standard procedure has been to resort to the "out-of-kilter" methods



which, although admirably efficient for general networks, were not originally designed to exploit the special topological properties of the transportation network. Recently, however, Graves and Thrall [32, p. 272] have specialized the out-of-kilter algorithm for the special properties of transportation networks.

In this paper we present a dual method based on those of [2,3,7] that operates directly on the transportation network, and which accommodates the fully capacitated problem (with finite and/or infinite upper and lower bounds). Our approach implicitly relies on the "poly-u" procedure in a manner analogous to that indicated by Charnes and Kirby for the uncapacitated case, but our development for the general capacitated problem is completely straightforward and requires no reference to "poly-u" methods for its justification. By coupling this specialization of the "poly-u" procedure with the predecessor and augmented predecessor index methods [18,19] for accelerating the determination of basis trees and dual evaluators a streamlined computer implementation is achieved.

We also specify a method that gives a dual feasible starting basis. Our "dual start" method requires an amount of computation somewhere between that of the "northwest corner" and "VAM" methods commonly used to obtain primal feasible starts. An important feature of our method, however, is its automatic avoidance of a starting basis containing "inadmissible" cells. As we show in Section 5, the "northwest corner" and "VAM" procedures may actually select such cells to be in a starting basis. Thus the basic solutions provided by these methods may well be "primal feasible" only in an artificial sense. The basic solutions provided by our method, however, are dual feasible without qualification. [A dual feasible basis may legitimately contain cells whose variables are constrained to zero, and hence "inadmissible" cells, but our method effectively bypasses them.] Because large scale transportation problems encountered in an industrial setting are typically quite sparse (i.e., contain a large number of inadmissible cells), the computational study in Section 6 examines the effect of problem density on solution time. [Density is equal to the number of arcs in the problem divided by the number of total possible arcs.]

## 2.0 NOTATION AND PROBLEM STATEMENT

We write the capacitated transportation problem in the form:

$$\text{Minimize } x_{00} = \sum_{\substack{i \in M \\ j \in N}} c_{ij} x_{ij} \quad (1)$$

$$\text{subject to } \sum_{j \in N} x_{ij} = a_i, \quad i \in M = \{1, 2, \dots, m\} \quad (2)$$

$$\sum_{i \in M} x_{ij} = b_j, \quad j \in N = \{1, 2, \dots, n\} \quad (3)$$

$$L_{ij} \leq x_{ij} \leq U_{ij}, \quad i \in M, j \in N \quad (4)$$

where the coefficients  $a_i$ ,  $b_j$ ,  $L_{ij}$  are finite integers, and  $U_{ij}$  are integers (if they are finite valued), and  $\sum_{i \in M} a_i = \sum_{j \in N} b_j$ . (Ways for casting a

variety of network problems in this formulation are given in [6,9,30,32,35].)

Following standard terminology, the  $a_i$  parameters are called "supplies," and the  $b_j$  parameters are called "demands." We associate these supplies and demands, respectively, with the rows and columns of an  $m \times n$  transportation tableau whose cells contain the "cost coefficients"  $c_{ij}$ .

A set of  $m+n-1$  cells of the transportation tableau is a basis if each tableau row and column contains at least one of the cells and if no subset of these cells constitutes a cycle (i.e., has the form  $(i_1, i_2), (i_2, i_3), \dots, (i_p, i_1)$ ). A cell (and its associated variable  $x_{ij}$ ) is called basic if it is contained among those cells in the basis and is called nonbasic otherwise.

A basic solution is the unique assignment of the values to the  $x_{ij}$  variables satisfying the equations (2) and (3) that results once each nonbasic  $x_{ij}$  has been set equal to  $L_{ij}$  or equal to  $U_{ij}$  (provided the value of the relevant bound is finite). If such a solution satisfies (4) for all of the variables, then it is called primal feasible.

Corresponding to a particular basis is a set of "row multipliers"  $R_i$  and a set of "column multipliers"  $K_j$  (not unique) such that the "updated cost coefficient"  $\pi_{ij}$ , defined by  $\pi_{ij} = R_i + K_j - c_{ij}$ , is zero for all basic cells; a basic solution is dual feasible if in addition  $\pi_{ij} \leq 0$  for all nonbasic variables  $x_{ij}$  set equal to  $L_{ij}$  and  $\pi_{ij} \geq 0$  for all nonbasic variables  $x_{ij}$  set equal to  $U_{ij}$ . (The multipliers  $R_i$  and  $K_j$  on which the  $\pi_{ij}$  are based represent values assigned to the variables of the dual of the transportation problem.) By fundamental linear programming theory, a basic solution that is both primal and dual feasible is optimal for the transportation problem.



### 3.0 THE DOUBLE PRICING ALGORITHM

To lay the foundation for specializing the Lemke dual method to the capacitated transportation problem, we first review the steps of the dual method in a general (bounded variable) minimization linear programming framework. (See Footnote 1.)

1. Begin with a dual feasible solution.
2. Select a basic variable (call it  $y_r$ ) that violates either its upper or lower bound. (If no such variable exists, the current basic solution is optimal and the method stops.)

3. Determine the unique "updated" linear equation which expresses the selected basic variable  $y_r$  as a linear combination of the current nonbasic variables; i.e.,  $y_r = \lambda_0 + \sum_{k \in \text{NB}} \lambda_k (-y_k)$ , where NB denotes the index set for the current nonbasic variables  $y_k$ .

4. Let  $\lambda'_k = -\lambda_k$  for  $k \in \text{NB}$  if  $y_k$  is set equal to its lower bound and  $y_r$  violates its lower bound, or if  $y_k$  is set equal to its upper bound and  $y_r$  violates its upper bound. For the remaining  $k \in \text{NB}$ , let  $\lambda'_k = \lambda_k$  and let  $\text{NB}^+ = \{j \in \text{NB} : \lambda'_j > 0\}$ . If  $\text{NB}^+$  is empty, then the problem has no primal feasible solution and the algorithm stops.

5. Identify the unique "updated" equation that expressed the objective function variable to be minimized (call it  $y_0$ ) in terms of the current nonbasic variables; i.e.,

$$y_0 = \pi_0 + \sum_{k \in \text{NB}} \pi_k (-y_k)$$

6. Identify a nonbasic variable  $y_s$ ,  $s \in \text{NB}^+$ , such that

$$|-\pi_s/\lambda_s| = \min_{k \in \text{NB}^+} \{ |-\pi_k/\lambda_k| \}$$

$$(\text{For } k \in \text{NB}^+, |-\pi_k/\lambda_k| = |\pi_k| / \lambda'_k.)$$

7. Determine a new current basic solution by removing  $s$  and adding  $r$  to NB (making  $y_s$  basic and  $y_r$  nonbasic), and assigning  $y_r$  the value of the bound it previously violated while holding the other nonbasic variables constant (identifying the values thus assigned to the new set of current basic variables). Then return to Instruction 2.

Finiteness of the foregoing method is assured through the use of "perturbation" or "lexicographic" schemes (see [5,9]). Such schemes will not be discussed here.

To refine the general dual method to the capacitated transportation problem, it is important to note that the "row-column sum (also called the MODI method [9]) method" [6, vol. II] for solving the transportation problem with the primal simplex method can be applied to determine the  $\pi_k$  values of Instruction 5 (which correspond to the  $\pi_{ij}$  values indicated in the preceding section), and to effect the "basis exchange" step of Instruction 7. Thus, to complete the specification of the dual method specialized to the transportation problem we require a procedure for determining the  $\lambda_k$  coefficients introduced in Instruction 3 (and referred to in Instructions 4 and 6).

The close resemblance of the goals of determining the  $\pi_k$  and  $\lambda_k$  coefficients when the dual method is stated as above suggests that the procedures for achieving these goals should likewise be similar. Indeed, the so-called "pricing-out" procedure for determining the  $\pi_k$  values for the transportation problem directly extends to a "double pricing-out" procedure for joint determination of both the  $\pi_k$  and  $\lambda_k$  values.

In the transportation context, we shall denote the basic variable  $x_r$  selected in Instruction 2 by  $x_{pq}$ , and represent the equation of instruction 3 by

$$x_{pq} = \lambda_{00} + \sum_{\substack{i \in M \\ j \in N}} \lambda_{ij} (-x_{ij})$$

(where  $\lambda_{ij} = 0$  if  $x_{ij}$  is basic). The justification of the following result is then immediately apparent from our foregoing remarks.

Lemma:

Let  $c'_{pq} = 1$  and  $c'_{ij} = 0$  for all  $i \in M$ ,  $j \in N$ , and  $(i, j) \neq (p, q)$ . If  $R'_i$  and  $K'_j$  are row and column multipliers such that  $c'_{ij} = R'_i + K'_j$  for all basic  $x_{ij}$ , then the  $\lambda_{ij}$  values are given by

$$\lambda_{ij} = R'_i + K'_j - x'_{ij}, \quad i \in M, \quad j \in N.$$

Proof:

From the "row-column sum" method, the unique "updated" equation that expresses the objective function variable to be minimized  $x_{00} = \sum_{\substack{i \in M \\ j \in N}} c_{ij} x_{ij}$



as a linear combination of the current nonbasic variables is  $x_{00} = \pi_{00} + \sum_{\substack{i \in M \\ j \in N}} \pi_{ij} (-x_{ij})$  where  $\pi_{ij} = R_i + K_j - c_{ij}$ ,  $i \in M$ ,  $j \in N$  and  $R_i$  and  $K_j$  are row and column multipliers such that  $c_{ij} = R_i + K_j$  for all basic  $x_{ij}$ .

For the costs  $c'_{ij}$  defined above,  $x_{00} = \sum_{\substack{i \in M \\ j \in N}} c'_{ij} x_{ij} = x_{pq}$ . Consequently,

$$\pi'_{00} + \sum_{\substack{i \in M \\ j \in N}} \pi'_{ij} (-x_{ij}) = \lambda_{00} + \sum_{\substack{i \in M \\ j \in N}} \lambda_{ij} (-x_{ij})$$

where  $\pi'_{ij} = R'_i + K'_j - c'_{ij}$ ,  $i \in M$ ,  $j \in N$ , and  $R'_i + K'_j - c'_{ij} = 0$  for all basic  $x_{ij}$ . Thus,

$$\lambda_{ij} = R'_i + K'_j - c'_{ij}, \quad i \in M, \quad j \in N.$$

As intimated earlier, this result constitutes a specialization of the Charnes poly- $\pi$  procedure. A thorough discussion of the poly- $\pi$  procedure in a general setting is unnecessary to see the validity of the result in the present application and will, therefore, be omitted. (See[6,7].)

It is also clear that the "row-column sum" rules for determining the  $R_i$  and  $K_j$  multipliers (that yield the  $\pi_{ij}$ ) can be applied to determine the  $R'_i$  and  $K'_j$  multipliers and the  $R_i$  and  $K_j$  multipliers simultaneously. Moreover, the topological structure of the transportation problem, namely the triangularity property of a basis, permits the conclusion that  $\lambda_{ij} = -1, 0$ , or  $1$  for all  $i \in M$ ,  $j \in N$ .

Also, using only the basis information the  $\lambda_{ij}$  values can be easily determined with a minimum amount of work, thus alleviating the need to update the  $\lambda_{ij}$  values from iteration to iteration. Further the augmented predecessor index method [19] provides a concise and efficient list structure for finding the  $R'_i$  and  $K'_j$  multipliers. The computer implementation of this algorithm thus uses this list structure. Unfortunately, even with all these simplifications a great deal of the total computational time is spent determining the  $\lambda_{ij}$  values and the  $\pi_{ij}$  values used to find the next pivot. This is due to the large number of these values to be found per pivot; for instance, if the problem has 10,000 variables then 10,000  $\lambda_{ij}$  values must be determined on each iteration.

#### 4.0 COMPLETED SPECIALIZATION

To bring our foregoing comments together and make them specific, we now indicate the full specialization of the dual method of Section 3 to the capacitated transportation problem in the following detailed set of instructions:

1. Begin with a dual feasible basic solution, denoted by  $x_{ij} = x^*_{ij}$ . (Section 5 presents an algorithm for obtaining such a solution.)
2. Select a basic variable  $x_{pq}$  such that  $x^*_{pq} > U_{pq}$  or  $x^*_{pq} < L_{pq}$ . (If no such variable exists, the solution  $x_{ij} = x^*_{ij}$  for  $i \in M, j \in N$  is optimal and the method stops.)

3. Determine the  $\pi_{ij}$  and  $\lambda_{ij}$  values simultaneously by the following rules for calculating the multipliers  $R_i, R'_i, K_j$ , and  $K'_j$ . To start, let  $R_p = R'_p = 0, K_q = c_{pq}, K'_q = 1$  and create the sets  $P = \{p\}$  and  $Q = \{q\}$ . In general, if there exists a basic variable  $x_{ij}$  such that  $i \in P$  and  $j \in N - Q$ , then let  $K_j = c_{ij} - R_i, K'_j = -R'_i$ , and let  $Q = Q \cup \{j\}$ . Similarly, if there exists a basic variable  $x_{ij}$  such that  $j \in Q$  and  $i \in M - P$ , then let  $R_i = c_{ij} - K_j, R'_i = -K'_j$ , and let  $P = P \cup \{i\}$ . Repeat the foregoing until  $P = N$  and  $Q = M$  and determine the values  $\pi_{ij} = R_i + K_j - c_{ij}$  and  $\lambda_{ij} = R'_i + K'_j - c'_{ij}$  for all  $i \in M, j \in N$  where  $c'_{ij} = 0, i \in M, j \in N$ , and  $(i,j) \neq (p,q)$  and  $c'_{pq} = 1$ . [The  $\pi_{ij}$  values will initially be known from the algorithm of Section 5 for obtaining a basic dual feasible start and therefore need not be calculated on the first iteration of this step.]

4. Let  $\delta$  be a "flag" set which is equal to  $-1$  if  $x^*_{pq} < L_{pq}$  and which is set equal to  $+1$  if  $x^*_{pq} > U_{pq}$ . Then let  $NB^+$  be the set of cells  $(i,j)$  such that either  $\lambda_{ij} = \delta$  and  $x^*_{ij} = L_{ij}$ , or  $\lambda_{ij} = -\delta$  and  $x^*_{ij} = U_{ij}$ . (If  $NB^+$  is empty, the transportation problem has no feasible solution and the algorithm stops.)

5. Identify a nonbasic variable  $x_{uv}, (u,v) \in NB^+$ , such that

$$|\pi_{uv}| = \min_{(i,j) \in NB^+} \{|\pi_{ij}|\}.$$

6. Determine the unique cycle created by adding the cell  $(u,v)$  to the cells of the basis (e.g., using the predecessor index method of [18]). Let

$\theta = L_{pq} - x_{pq}^*$  if  $x_{pq}^* < L_{pq}$  and otherwise, let  $\theta = U_{pq} - x_{pq}^*$ . Beginning with cell  $(p,q)$ , let the new value of  $x_{ij}^*$  be  $x_{ij}^* + \theta$  for each odd cell of the cycle and let the new value of  $x_{ij}^*$  be  $x_{ij}^* - \theta$  for each even cell of the cycle. Designate  $x_{uv}$  basic,  $x_{pq}$  nonbasic, and return to Instruction 2.

#### 5.0 ALGORITHMS FOR FINDING A BASIC DUAL FEASIBLE SOLUTION

No general procedure exists for obtaining a starting primal basic feasible solution for the capacitated transportation problem or for the uncapacitated transportation problem when some of the cells are blocked out (inadmissible). Thus, when solving such problems using a primal simplex approach, an artificial starting basis usually must be employed. One of the major advantages of this algorithm is that it is always possible to find a starting basic feasible solution for the dual of such problems. Furthermore, this dual start procedure provides a basic dual feasible solution for a network if the network is rewritten as an equivalent transportation problem.

To the best of the authors' knowledge, no other algorithms which have employed the dual method [2,3,7] or related "primal-dual" algorithms [13,14,15] exploit the topological structure of the dual to a transportation problem to provide a dual feasible start.

The starting method we propose may be described very simply as follows:

1. To start, set  $R_1 = 0$ . Create the set  $P = \{1\}$ .
2. Let  $T$  be the set of all admissible cells in the transportation problem (i.e.,  $T$  contains all the unblocked cells in the transportation problem.). Set  $K_j = c_{1j}$  for all  $(1,j) \in T$ . Create the set  $Q = \{j: K_j = c_{1j} \text{ for all } (1,j) \in T\}$  and the set  $B = \{(1,j): K_j = c_{1j} \text{ for all } (1,j) \in T\}$ .
3. Let  $Q_1 = \{j: j \in Q \text{ and } (1,j) \in T\}$ . For each  $i \in M - P$  such that  $Q_i$  is nonempty, identify an index  $j^* \in Q_i$  for which  $c_{ij^*} - K_{j^*} = \min_{j \in Q_i} \{c_{ij} - K_j\}$ , and set  $R_i = c_{ij^*} - K_{j^*}$ ,  $P = P \cup \{i\}$ , and  $B = B \cup \{(i, j^*)\}$ .
4. Let  $P_j = \{i: i \in P \text{ and } (i,j) \in T\}$ . For each  $j \in N - Q$  such that  $P_j$  is nonempty, identify an index  $i^*$  for which  $c_{i^*j} - R_{i^*} = \min_{i \in P_j} \{c_{ij} - R_i\}$  and set  $K_j = c_{i^*j} - R_{i^*}$ ,  $Q = Q \cup \{j\}$  and  $B = B \cup \{(i^*, j)\}$ .
5. Continue executing Instructions 3 and 4 until  $M = P$  and  $N = Q$ .



Lemma:

When Instructions 1 - 5 are completed, the cells in B form a basis and the multipliers  $R_i$  and  $K_j$  defined above satisfy the following properties:

1.  $R_i + K_j = c_{ij}$  for all  $(i,j) \in B$ .
2.  $R_i + K_j - c_{ij} \leq 0$  for all  $(i,j) \in T$ .

Proof:

The method will continue to execute Instructions 3 and 4 until  $P = M$  and  $Q = N$  provided that  $Q_i$  and  $P_j$  are not both empty at some iteration. Assume, however, that both  $Q_i$  and  $P_j$  are empty at some iteration, then the problem has no basic solution except by using an inadmissible cell. This follows by noting that the stated conditions imply that the problem's admissible cells are all found among those cells  $(i,j)$  for  $i \in P$  and  $j \in Q$  or for which  $i \in M-P$  and  $j \in N-Q$  and hence there is no connected set of admissible cells that spans all rows and columns. Thus assuming that the transportation problem is connected, the algorithm will continue to execute until  $P = M$  and  $Q = N$ .

When the algorithm stops, B contains  $m+n-1$  cells since one new cell is added to B each time an  $R_i$  or a  $K_j$  is determined except for  $R_1$ . The cells of B clearly span all the rows and columns. Further, the cells in B contain no cycles. To verify this, assume the contrary and let  $(r,s)$  be the first cell added to B which creates a cycle with the previous cells. Then there must be a cell  $(r,j) \in B$  and a cell  $(i,s) \in B$ . But this is impossible when  $(r,s)$  is added, since all cells  $(i,j) \in B$  satisfy  $i \in P$  and  $j \in Q$ , whereas to augment B with the cell  $(r,s)$  either  $r \in M-P$  (Step 3) or  $s \in N-Q$  (Step 4). Thus B forms a basis.

Finally, the relations 1 and 2 are in immediate consequence of the definition of the  $R_i$  and  $K_j$  values. Thus, the proof of the lemma is complete.

[We remark that the foregoing proof also justifies a more flexible version of the dual starting method in which Step 2 is applied to only a single cell  $(1,j) \in T$ , and Steps 3 and 4 are executed in any desired sequence, selecting only one index  $i$  or  $j$  in these steps in any single execution. Of course, the method can also begin with a row index other than  $i = 1$ .]

From relations 1 and 2, the lemma implies that a solution obtained by setting the flow of each nonbasic cell equal to its lower bound  $L_{ij}$  and each basic cell equal to the unique flow specified by the basis (once the nonbasic cells have been set) will yield a basic dual feasible solution. Furthermore, the lemma implies that such a basic dual feasible solution may be obtained for any capacitated transportation problem as long as it is connected.

To illustrate the algorithm, consider the uncapacitated transportation problem in Tableau 1. The numbers in the upper left hand corner of each cell indicate the cost of that cell. (If the cell is blocked out, it is assigned a cost of M.)

$R_1 = 0$   
 $R_2 = -2$   
 $R_3 = -4$   
 $R_4 = 9$

$K_1 = 1 \quad K_2 = -4 \quad K_3 = -1 \quad K_4 = 5$							
1	*	M		M		5	*
	16						6
2		2		5		3	*
							8
10		7		2		1	*
							6
10	*	5	*	8	*	M	
	4		8		8		
	12		8		8		8

10  
8  
6  
12

Tableau 1

The  $R_i$  and  $K_j$  values indicated alongside the tableau are determined by instructions 1 - 5. The cells with an asterisk (\*) in the upper right hand corner are the basic cells. The circled numbers in these cells indicate the unique shipping amounts specified by the basis.

This example can be further used to illustrate the fact that employing the usual techniques for obtaining a starting basic primal feasible solution may yield an artificial start.

In particular, applying the northwest corner rule and the Vogel's Approximation method, [30], respectively, to the transportation problem in Tableau 1 gives the artificial bases of Tableaus 2 and 3 below. (The circled entries indicate the shipping amounts.)

1		M		M		5		10
	10							
2		2		5		3		8
	2		6					
10		7		2		1		6
			2		4			
10		5		8		M		12
					4		8	

Tableau 2

1		M		M		5		10
	10							
2		2		5		3		8
	2		6					
10		7		2		1		6
							6	
10		5		8		M		12
			2		8		2	

Tableau 3

## 6.0 CODE DEVELOPMENT AND COMPUTATIONAL COMPARISON

### 6.1 OVERVIEW

This section presents a computational comparison of a code based on the above developments with two widely used out-of-kilter production codes and a state of the art large scale LP code, OPHELIE/LP. We also examine the effect of problem density on solution time, where density is equal to the number of arcs (cells) in the problem divided by the number of total possible arcs. (The interest in density is stimulated by the fact that large "real world" transportation problems are quite sparse.)

The two out-of-kilter codes which we tested are those of SHARE and Boeing. The SHARE code was written by R.J. Clasen of the RAND Corporation and is available for general distribution [8,29]. The Boeing code, which was obtained through Chris Witzgall, was developed at the Boeing research laboratories. Both of these codes and the dual code are in-core codes: i.e., the program and all of the problem data simultaneously reside in

fast-access memory. All three codes are written in FORTRAN and none of them have been tuned (optimized) for a particular compiler. All of the codes were run on the CDC 6600 (which has a maximum memory of 130,000 words) at the University of Texas Computation Center using the RUN compiler. The computer jobs were executed during periods when the machine load was approximately the same and all solution times are exclusive of input and output; i.e., the total time spent solving the problem was recorded by calling a Real Time Clock on starting to solve the problem and again when the solution was obtained.

The general simplex linear programming computer code employed in the study was Control Data's OPHELIE/LP code. OPHELIE/LP is a subsystem of the OPHELIE II Mathematical Programming System which is programmed to exploit the characteristics of the CDC 6600 computer.

To guarantee a comprehensive comparison of the procedures under analysis, the transportation problems used in the study varied between .5 percent and 90 percent dense and varied in size from 10 x 10 to 500 x 500 (origins x destinations). A total of 65 different uncapacitated transportation problems were examined, all of which were randomly generated using a uniform probability distribution. The total supply of each  $m \times m$  transportation problem was set equal to 1000  $m$  and the supply and demand amounts were picked using a uniform probability distribution between 0 and 2000. The only other restrictions placed on the problems consisted of requiring the number of variables to be less than or equal to 10,000 and requiring the cost coefficients to lie between 1 and 100.

## 6.2 DOUBLE PRICING DUAL CODE DEVELOPMENT

The computer code embodying the ideas of the preceding section was written in FORTRAN IV and tested on a CDC 6600 with a maximum memory of 130,000 words. To solve a problem with  $m$  origins,  $n$  destinations, and  $r$  admissible cells (without exploiting the word size of the machine) this in-core code requires  $3r + 19m + 17n + 20,000$  words. It would have been possible by exploiting the fact that the costs are integer-valued, to store more than one cost coefficient per word and in this manner solve much larger problems. However, our purpose was to develop a code whose capabilities did

not depend on the unique characteristics of a particular computer (e.g., word size, etc.). To permit the solution of large problems we organized the code to utilize a "packing" scheme which stores only the "real" costs of the transportation matrix (i.e., only the costs of the admissible cells). Thus for a 200 x 200 problem, with 10 percent density, this approach stores only 4,000 of the 40,000 elements stored by standard schemes. The more economical storage scheme incurs a time disadvantage in packing and unpacking cost coefficients, but materially reduces the number of updated costs that have to be computed for low density problems in step 3 of section 4.

The program consists of a main program and fifteen subroutines, and may be conceptually depicted as in boxes 1 - 5 in Figure 1. Subdividing the program into many different subroutines made it possible to test numerous variations without extensive recoding. However, this subdivision inevitably slowed the code somewhat by requiring the computer to process subroutine calls and returns rather than jump instructions.

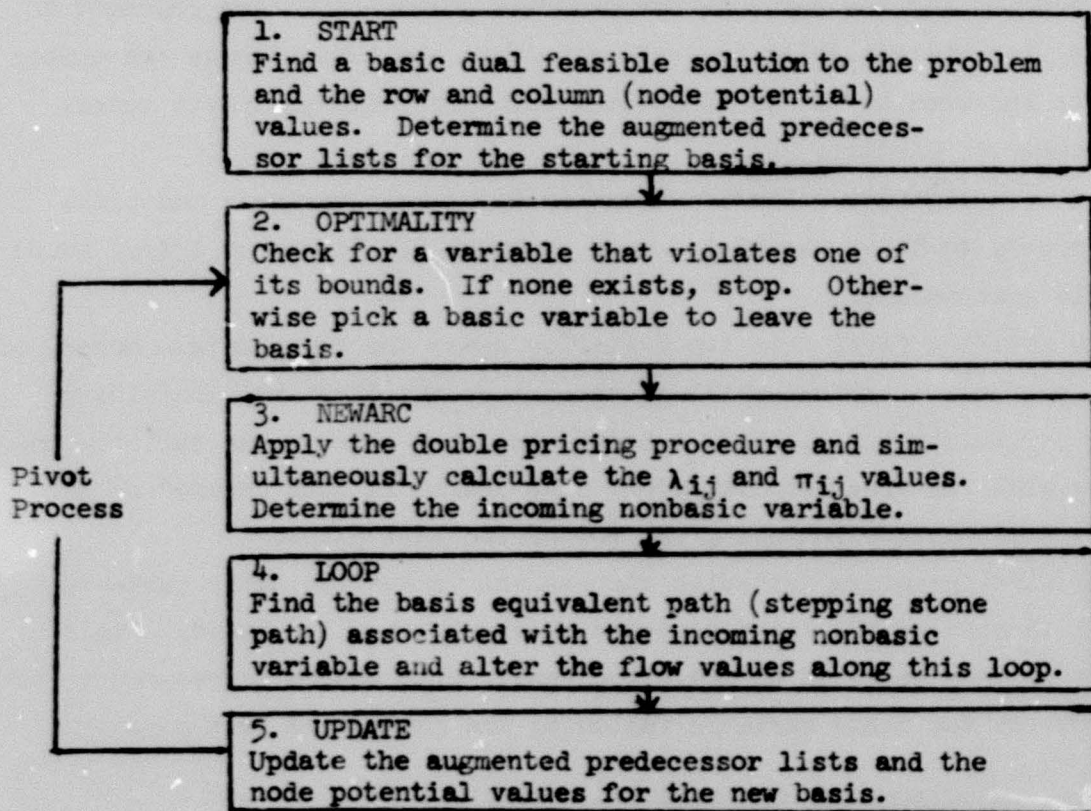


Figure 1 - Flow Diagram for the Dual Transportation Code



The total time spent in each of the boxes 1 - 5 was recorded by calling a Real Time clock upon entering and leaving each of these functions. A count was also made of the total number of pivots performed. In Tables 1 and 2, we report median values for the total solution time, the start time (time spent in box 1), the number of pivots, the total pivot time (total time spent in boxes 2 - 5), and the average pivot time (total pivot time divided by the number of pivots). In addition, Table 1 contains the total time spent finding the non-basic arc to enter the basis (total time spent in box 3) and Table 2 contains the range of solution times.

In developing the code only the dual start procedure developed in Section 5 was tested. (Subsequent to our original development of this dual start procedure, we developed other dual start procedures which appear in [17].)

Three different criteria were tested for picking the basic variable to leave the basis. These criteria were the first negative criterion, modified first negative criterion, and most negative criterion. The relevant tradeoffs for the basis change criteria involve the time consumed in searching for the variables to enter and leave the basis versus the number of pivots required to obtain an optimal solution (time per pivot versus total number of pivots).

The most negative criterion examines each basic variable and picks that variable to leave the basis which violates its (upper or lower) bound by the largest amount.

The modified first negative criterion scans the rows (origin nodes) of the transportation tableau until it encounters the first row containing a basic variable which violates a bound, and then selects the variable in this row with the greatest violation. The search is then resumed on the next pivot in the row following the row of the last pivot.

The first negative criterion successively scans the basic variables until it encounters the first variable which violates its bounds. This variable is then selected to leave the basis. The search is resumed on the next pivot at the basic variable following the one of the last pivot.

TABLE 1  
Total Solution Time Relative to Basis Change Criteria  
for 250 x 250 and 500 x 500 Transportation Problems

250 x 250 Transportation Problems

MOST NEGATIVE CRITERION

DENSITY	Solution Time	Start Time	Pivot Time	No. of Pivots	NEWARC*	Time/Pivot
.010	18.604	1.451	17.153	410	8.580	.042
.013	45.624	1.853	44.771	900	24.792	.050
.017	52.781	1.870	51.911	975	30.369	.053
.020	60.575	1.332	59.243	983	37.557	.060
.023	66.401	1.628	64.773	992	43.631	.065

MODIFIED FIRST NEGATIVE CRITERION

.010	18.315	1.421	16.894	461	8.745	.037
.013	38.499	1.812	36.687	861	20.787	.043
.017	40.690	1.841	38.849	854	23.460	.045
.020	43.184	1.429	41.755	818	26.852	.051
.023	59.865	1.719	58.146	1053	40.163	.055

FIRST NEGATIVE CRITERION

.010	20.671	1.425	19.246	595	12.358	.032
.013	22.889	1.843	22.046	852	15.473	.026
.017	31.311	1.845	29.466	1005	21.145	.029
.020	37.269	1.396	35.873	1074	26.924	.033
.023	35.872	1.675	34.197	907	26.775	.038

500 x 500 Transportation Problems

MOST NEGATIVE CRITERION

.03	151.743	3.013	147.730	1741	70.314	.085
.04	175.108	3.794	171.314	1866	86.636	.092

MODIFIED FIRST NEGATIVE CRITERION

.03	142.288	2.916	139.372	2126	78.500	.066
.04	155.707	3.860	151.847	2147	85.058	.092

FIRST NEGATIVE CRITERION

.03	107.599	2.944	104.655	2253	73.477	.046
.04	111.905	3.927	107.978	2188	77.257	.049

\*Time to find the new arc entering the basis.

15-a

Table 1 contains the results of testing these criteria on the 250 x 250 and 500 x 500 transportation problems. Not all the 500 x 500 transportation problems were solved due to the fact that our results strongly indicated that the first negative criterion was best for this problem size. This was quite surprising since similar tests for the primal transportation algorithm [10,20,33] have shown the modified first negative criterion to be preferred.

Table 1 shows that more pivots are required on the 500 x 500 problems as the simplicity of the criteria increases. This is not invariably the case for the 250 x 250 problems. For the .013 dense, 250 x 250 problem the number of pivots is larger for the most negative and modified first negative criteria than for the first negative criterion. Specifically, the superiority of the first negative criterion strictly improves as density increases. This seems somewhat peculiar since a change in density does not affect the number of basic variables. A partial explanation of this result is provided by studying the "NEWARC" column. More precisely, observe that the "NEWARC" times are much larger for the most negative and modified first negative criteria than for the first negative criteria. This indicates that the basic variable picked by the former criteria have significantly more negative  $\lambda_{ij}$  values associated with it than does the basic variable picked by the latter criterion. This conclusion is occasioned by the fact that the updated costs associated with non-negative  $\lambda_{ij}$  values do not have to be calculated. (Recall these problems are uncapacitated.) In addition, the minimization of the relevant updated costs is over a smaller set. All in all this reflects an interesting topological property of the basic arcs

NEWARC/number of pivots).

Primarily Table 1 indicates that the most negative and modified first negative criteria have little to offer by comparison with the first negative criterion since they require more search time to find the basic variable to leave the basis, more computation to determine the non-basic variable to enter the basis, and they do not substantially reduce the number of pivots.

Table 1 indicates that as problem size and/or density increases "NEWARC" time grows disproportionately large. For this reason the dual method is not an effective method for solving large problems.

### 6.3 COMPUTATION COMPARISON

The results of Table 2 demonstrate that the graph structure of the transportation problem does merit the use of special purpose algorithms. The dual code and the out-of-kilter codes are at least 20 times faster than the OPHELIE/LP general simplex linear programming computer code.

The median times for the OPHELIE/LP code were derived after first conducting a number of trial runs to determine the best procedure for selecting the next incoming variable. The trial runs indicated that 30 variables should be considered at a time and, of these, the variable associated with most negative updated cost should be selected.

The data in Table 2 also indicate that neither the SHARE code nor the Boeing code dominates the other with respect to solution time. The solution times of the Boeing code, however, appear to be more erratic, with a somewhat broader range. The solution times of the dual code also appear to be more erratic than the SHARE times. This characteristic of the dual code and the Boeing code reflects their dependence on density. Reduced density, as might be expected, leads to definite improvements in computation times.

The most important finding (and most disappointing from the standpoint of the developments of Sections 3 and 4) is the indication in Table 2 that the dual code falls behind the others for problems of size 250 x 250 and larger, although it is decidedly faster than the out-of-kilter codes on small to moderate size problems (up through 150 x 150). The primary cause of this degeneration of superiority is the large amount of time required to find the variable to enter the basis. Since our modification of the dual algorithm was in fact designed to minimize these calculations, we conclude that the dual method is not a competitive "dead start" procedure for solving large problems - i.e., its potential value for large problems lies in those situations in which an extremely good dual starting solution is available. Further, it is unlikely that any minor modifications which are derived in the future will alter this conclusion. This conclusion is based upon several factors:

TABLE 2  
Median Solution Times (sec) for Out-of-Kilter,  
OPHELIE/LP, and Dual Codes

Problem Size	Density <sup>2</sup>	OPHELIE/LP			SHARE			Boeing			Dual		
		Solution <sup>1</sup> Time	Range	Solution <sup>1</sup> Time	Solution <sup>1</sup> Time	Range	Solution <sup>1</sup> Time	Solution <sup>1</sup> Time	Range	Solution <sup>1</sup> Time	Start Time	No. Pivots	Total Pivot Time
10 x 10	.31	.75	.50-1.05	.10	.05-.14	.08	.04-.15	.02	.01-.03	.01	.01	8	.0027
20 x 20	.66	4.01	3.21-5.62	.68	.64-.87	1.04	.58-2.36	.22	.17-.51	.02	.02	37	.005
30 x 30	.61	DNR		2.04	1.21-2.80	1.75	1.13-11.75	.68	.45-1.41	.05	.05	70	.009
40 x 40	.36	39.37	27.31-55.62	2.42	2.11-3.57	2.67	1.79-10.33	.90	.75-1.52	.05	.05	83	.010
50 x 50	.53	DNR		5.70	4.22-10.56	12.73	2.65-31.46	3.05	2.51-4.34	.10	.10	144	.020
60 x 60	.20	DNR		5.28	4.41-12.30	12.91	2.95-29.12	3.02	2.75-9.71	.09	.09	201	.015
70 x 70	.28	DNR		9.46	6.73-18.48	17.39	2.97-6.96	7.02	5.55-14.47	.14	.14	293	.021
80 x 80	.31	DNR		22.10	11.89-25.51	18.38	12.72-90.70	9.82	6.151-19.92	.19	.19	324	.030
90 x 90	.28	DNR		26.35	13.27-32.88	21.68	11.2-73.0	15.18	8.39-27.86	.22	.22	446	.034
100 x 100	.20	276.90	215.71-391.51	21.17	16.81-28.78	45.03	9.91-79.80	14.62	10.27-39.51	.23	.23	476	.030
150 x 150	.23	DNR		56.20	46.98-88.01	535.16	49.2-1164.0	49.10	44.71-93.91	.75	.75	961	.050
250 x 250	.017	DNR		25.53	15.53-21.74	15.90	13.17-16.91	31.31	23.67-40.26	1.04	1.04	1005	.029
500 x 500	.05	DNR		85.62	83.57-104.54	107.11	86.23-120.87	111.90	107.59-147.27	3.92	3.92	2188	.049

<sup>1</sup>All times are median times with five problems per group  
<sup>2</sup>Mean Density

DNR - Did not run.



1. Srinivasan and Thompson [33] found that the augmented predecessor index method [19] was twice as fast as any of the other methods for traversing and updating a spanning tree. Thus the procedures for pricing-out, updating node potentials, and locating basis equivalence paths in our code are based upon what is known to be the most effective method for performing these operations.

2. The problems used in this study overlap the problems used in [20]. This study [20] discloses that a primal transportation algorithm (embodying the augmented predecessor index method) coupled with the Row Minimum start rule and a "modified row first negative evaluator" rule is at least 100 times faster than OPHELIE. Further this method's median solution time for 100 x 100 transportation problems on a CDC 6600 computer was 1.9 seconds and 17 seconds on 1000 x 1000 problems. Thus, the dual transportation method would have to be improved approximately 7-fold on the 100 x 100 problems to match the performance of the primal code. Pursuing the relationship between the computational results of [20] and this study, the following remarks apply:

- a) The primal approach is less sensitive to density than the dual and out-of-kilter codes.
- b) The dual start of Section 5 is quite efficient. It requires only one-tenth of the time that the Vogel Approximation Method requires to obtain an initial solution and requires only half as many pivots to reach optimality.
- c) The inferiority of the dual method to the primal method lies in the drastic difference in the time per pivot of the two methods. The dual method's time per pivot is 5-6 times larger than that required by the primal.

3. The problems used in this study also overlap the problems being used in [4]. The preliminary results of that study indicate that an improved version of the out-of-kilter method coupled with a state-of-the-art computer implementation is from 4 to 12 times faster than SHARE. Thus from this partial comparison of three state-of-the-art implementations of the three fundamental solution approaches to solving transportation problems it seems apparent that the dual approach is running somewhat behind the others. Interestingly, as

though in anticipation of this, the original developers of the dual transportation approach [2,3,7], suggested that it be used chiefly for post-optimality type analysis or in application to small problems. (For example, it might be a good algorithm to use in conjunction with certain cutting plane or implicit enumeration procedures [31] that restart from previously optimal bases.)

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### Footnotes

1. A precise statement of this method does not seem to exist in the literature except in the treatment of Wagner [34] which utilizes an explicit replacement of a bounded variable by its associated slack. However, the statement given here is easily inferred from standard considerations. In particular, see the excellent discussions of such "generalizations" and "variations" in Jewell [22].
2. This code was developed by F. Glover, D. Karney, and D. Klingman.

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